Homework 4

PHYS798C Fall 2025
Due Tuesday, 25 November, 2025

1 Little-Parks Experiment and Flux Quantization

Consider a thin superconducting film of thickness $d << \lambda$ deposited onto a cylindrical dielectric filament (like a human hair for example). The radius of the filament's cross section is R. At room temperature, the filament is placed in a longitudinal magnetic field and then cooled down to a temperature below T_c. Then the external field is switched off.

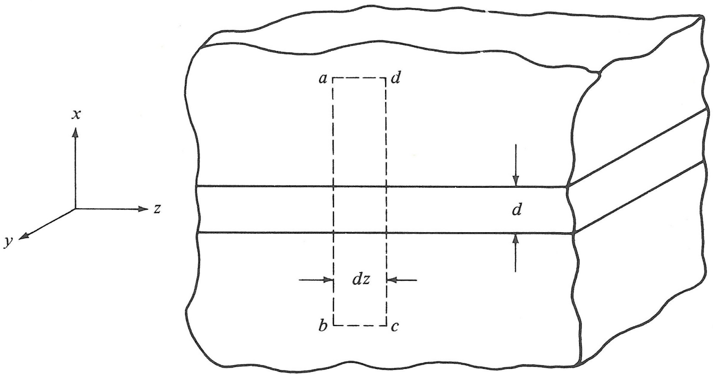
(a) Use fluxoid quantization in the thin cylindrical film to show that $m^* v_s(2\pi R) + q^*\Phi = nh$, where Φ is the trapped flux, n is a positive or negative integer or zero, and h is Planck's constant.

(b) Solve for the superfluid velocity v_s in terms of Φ/Φ_0 (recall that $\Phi_0 = h/e^*$). Plot v_s^2 vs. Φ/Φ_0 for 5 different choices of n centered on $n = 0$. This is effectively the kinetic energy of the supercurrent flow as a function of Φ/Φ_0.

(c) To minimize the kinetic energy as a function of Φ/Φ_0, the superconductor will choose different values of n (i.e. change the number of trapped flux) as the flux changes. This involves creating phase slips by suppressing T_c and allowing n to change by ± 1. Recall that the superconducting order parameter varies with v_s as $|\psi|^2 = |\psi_\infty|^2 \left(1 - (\frac{m^* \xi_{GL} v_s}{\hbar})^2\right)$. Show that the order parameter will be suppressed to zero when $\frac{1}{\xi_{GL}^2(T^*)} = \left(\frac{m^* v_s}{\hbar}\right)^2$, where T^* represents the suppresed T_c created by the screening supercurrent. Use the definition of the temperature-dependent GL coherence length $\xi_{GL}(T^*) = \xi_{GL}(0)/\sqrt{1-t^*}$, where $t^* = T^*/T_c$, to solve for (and plot) $\Delta T_c/T_c \equiv \frac{T^*-T_c}{T_c}$ in terms of $(n - \Phi/\Phi_0)^2$. These periodic variations in T_c vs. Φ/Φ_0 are measured in the Little-Parks experiment. *Hint*: see "The archived Little-Parks experiment lecture" under Lecture 18 in the Sup. Mat. on the class website.

2 Long Josephson Junctions

Consider a pair of identical semi-infinite superconductors sandwiching an insulating material of thickness d that supports Josephson tunneling. Consider the coordinate system and integration loop shown in the Figure. We will derive two wave equations for electromagnetic waves propagating in this parallel plate waveguide.



(a) Assume that the integration loops go deep enough into the superconductors such that the currents and fields go to zero for the segments $a-d$ and $b-c$ of length dz. Apply Faraday's law $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ on this loop (recall the integral form: $\oint_C \vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{S}$), and operate with $\partial/\partial z$ to derive the result

$$\frac{\partial^2 E_x^0}{\partial z^2}=-\left(\frac{2\lambda+d}{d}\right)\frac{\partial^2 B_y^0}{\partial t\partial z}$$

where the superscript ^{0} denotes quantities in the insulating layer.

(b) Take a similar integration loop normal to the $z-$axis and derive the result,

$$\frac{\partial^2 E_x^0}{\partial y^2}=\left(\frac{2\lambda+d}{d}\right)\frac{\partial^2 B_z^0}{\partial t\partial y}.$$

(c) Now operate with $\partial/\partial t$ on the $x-$component of Ampere's Law $\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$ to obtain the result

$$\frac{1}{\mu}\left(\frac{\partial^2 B_z^0}{\partial y\partial t}-\frac{\partial^2 B_y^0}{\partial z\partial t}\right)=\frac{\partial J_x}{\partial t}+\epsilon\frac{\partial^2 E_x^0}{\partial t^2}$$

(d) Now substitute the derivatives of B^0 from (a) and (b) to derive the result,

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)E_x^0=\frac{1}{\varepsilon v_{ph}^2}\frac{\partial J_x}{\partial t}$$

where $v_{ph}=\frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{d}{d+2\lambda}}$. Note that if $\frac{\partial J_x}{\partial t}=0$ then this equation describes TEM waves propagating down a superconducting waveguide, known as Swihart modes. The inductance of the waveguide is proportional to $L\sim d+2\lambda$ and the capacitance $C\sim 1/d$, hence $v_{ph}\sim\frac{1}{\sqrt{LC}}\sim\sqrt{\frac{d}{d+2\lambda}}$. As the penetration depth grows the phase velocity of the wave can be slowed significantly.

(e) Now assume that there is Josephson coupling between the two plates through the insulator. Taking the top plate as the positive potential we have from the second Josephson equation

$\frac{\partial\gamma}{\partial t}=\frac{2eV}{\hbar}=-\frac{2eE_x^0 d}{\hbar}$

Solving this for E_x^0 and using the first Josephson equation, derive the result

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\gamma(y,z)=\frac{\sin\gamma(y,z)}{\lambda_J^2},$$

where we have taken the $t=0$ time reference where $\gamma=0$ in the time integration and $\lambda_J^2\equiv\hbar/\left[2eJ_c\mu\left(2\lambda+d\right)\right]$ is the Josephson penetration depth. This is a form of the famous sine-Gordon equation. Estimate the Josephson penetration depth for a long junction with $2\lambda+d=90\ nm$ and $J_c=10^2 A/cm^2$.

(f) Consider a linearlized solution to the Josephson wave equation. Assume small phase variation across the junction as a function of time so that $\gamma(y,z,t)=\gamma_0(y,z)+$

$\gamma_1(y,z,t)$, where γ_0 is a spatially-dependent time average, and $\gamma_1 \ll \gamma_0$. With this substitution and assuming that $\cos\gamma_1 \simeq 1$, show that

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\gamma_1 = \left(\frac{\cos\gamma_0}{\lambda_J^2}\right)\gamma_1$$

Now assume that γ_0 is a constant. Assume a travelling wave solution of the form $\gamma_1 = e^{-i\left(\omega t - \vec{\beta}\cdot\vec{r}\right)}$ and find the dispersion relation $\omega^2 = \beta^2 v_{ph}^2 + \omega_p^2$ where the Josephson plasma frequency is defined as $\omega_p^2 = \left(\frac{v_{ph}}{\lambda_J}\right)^2 \cos\gamma_0$. Plot the dispersion relation for both the Swihart and Josephson modes. How can you tune the Josephson plasma frequency? In the limit $\beta = 0$ there is no rf magnetic field present and there is a periodic exchange of energy between the electric field and the Josephson coupling energy, in close analogy with cold plasma oscillations.